$P_3^m(t) = ?$ $m = 0, 1, 2, 3$

$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n}((t^2-1)^n)$

$P_n^k(t) = (1-t^2)^{k/2} \frac{d^k}{dt^k}(P_n(t))$

$P_3^0(t) = P_3(t) = \frac{1}{2^3 3!} \frac{d^3}{dt^3}((t^2-1)^3) =$

$= \frac{1}{8 \cdot 6}\left((t^4-2t^2+1)(t^2-1)\right)''' = \frac{1}{48}\left(t^6-2t^4+t^2-t^4+2t^2-1\right)''' =$

$= \frac{1}{48}(t^6-3t^4+3t^2-1)''' = \frac{1}{48}(6 \cdot 5 \cdot 4 \cdot t^3 - 3 \cdot 4 \cdot 3 \cdot 2 \cdot t^1) =$

$= \frac{5}{2}t^3 - \frac{3}{2}t = 2,5\,t^3 - 1,5\,t = t(2,5t^2-1,5) = \cos\theta(2,5\cos^2\theta - 1,5)$

$P_3^1(t) = (1-t^2)^{1/2} \frac{d}{dt}(P_3(t)) = (1-t^2)^{1/2}(7,5 \cdot t^2 - 1,5) = \sin\theta(7,5\cos^2\theta - 1,5)$

$P_3^2(t) = (1-t^2)^{2/2} \frac{d^2}{dt^2}(P_3(t)) = (1-t^2)(15\,t) = \sin^2\theta(15) \cdot \cos\theta$

$P_3^3(t) = (1-t^2)^{3/2} \frac{d^3}{dt^3}(P_3(t)) = (1-t^2)^{3/2} \cdot 15 = \sin^3\theta \cdot 15$

$$\begin{cases} u_t = a^2 \Delta_3 u & 0 \le r < b \quad 0 < \theta < \pi \quad 0 \le \varphi < 2\pi \quad t > 0 \\ u|_{t=0} = \frac{3}{2} \sin 2\theta \cos\varphi \cdot \dfrac{J_{5/2}\left(\frac{\mu_2^{(1)}}{b} r\right)}{\sqrt{r}} \\ u|_{r} = 0 \end{cases}$$

$\mu_2^{(1)}$ - первый полож. корень $J_{\frac{5}{2}}(x) = 0$

1) $u = V(r, \theta, \varphi) \cdot T(t)$

$$\frac{\Delta_3 V}{V} = \frac{T'}{a^2 T} = -\lambda$$

$$\Delta_3 V = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2} \underbrace{\left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2 V}{\partial\varphi^2}\right)}_{\Delta_{\varphi,\theta} V}$$

2) $V = W(\theta, \varphi) \cdot R(r)$

$$\lambda W R + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 R') W + \frac{1}{r^2} \Delta_{\varphi,\theta} W \cdot R = 0$$

$$\cancel{\frac{1}{r^2}} \quad \frac{\frac{\partial}{\partial r}(r^2 R')}{R} + \underbrace{\lambda r^2}_{=+\mu} + \underbrace{\frac{\Delta_{\varphi,\theta} W}{W}}_{=-\mu} = 0$$

3) $$\begin{cases} \Delta_{\varphi,\theta} W + \mu W = 0 \\ W(\theta, \varphi) = W(\theta, \varphi + 2\pi) \end{cases}$$

$$W(\theta, \varphi) = \Theta(\varphi) \cdot \Phi(\varphi)$$

$$\frac{\Phi}{\sin\theta} \frac{\partial}{\partial\theta}(\sin\theta\, \Theta') + \frac{\Theta}{\sin^2\theta} \cdot \Phi'' + \mu \Theta \Phi = 0$$

$$\frac{\sin\theta}{\Theta}(\sin\theta\, \Theta')' + \mu \sin^2\theta + \underbrace{\frac{\Phi''}{\Phi}}_{=-\nu} = 0$$

4) $\Phi'' + \nu \Phi = 0, \quad \Phi(\varphi) = \Phi(\varphi + 2\pi)$

$\nu_0 = 0 \quad \Phi_0 = 1$

$\nu_k = k^2 \quad \Phi_k = a_k \sin k\varphi + b_k \cos k\varphi$

5) $$\frac{1}{\sin\theta} \frac{d}{d\theta}(\sin\theta \cdot \Theta') + \left(-\frac{k^2}{\sin^2\theta} + \mu\right) \Theta = 0$$

$t = \cos\theta \qquad \Theta(t(\theta))$

$$\frac{d\Theta}{d\theta} = \frac{d\Theta}{dt} = \frac{dt}{d\theta} = -\sin\theta \frac{d\Theta}{dt}$$

$$\frac{d}{d\theta}(\sin\theta \cdot \Theta'_\theta) = \frac{d}{d\theta}\left(-\sin^2\theta \frac{d\Theta}{dt}\right) = \frac{d}{d\theta}\left((t^2 - 1) \frac{d\Theta}{dt}\right) = \frac{d}{dt}\left((t^2-1)\frac{d\Theta}{dt}\right) \cdot (-\sin\theta)$$

$$\frac{d}{dt}\left((1-t^2)\frac{d\Theta}{dt}\right)+\left(\mu-\frac{k^2}{1-t^2}\right)\Theta=0$$

~ тип 2

$\mu_n=n(n+1)$

$\Theta=P_n^k(t)$

$\Rightarrow W=P_n^k(\cos\theta)(a_{nk}\cos k\varphi+b_{nk}\sin k\varphi)$

б) $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\lambda r^2R-\mu R=0$

$R=r^{-1/2}\,y(r)$

$R'=-\frac{1}{2}r^{-3/2}y+r^{-1/2}y'$

$r^2R'=-\frac{1}{2}r^{1/2}y+r^{3/2}y'$

$\frac{d}{dr}(r^2R')=-\frac{1}{4}r^{-\frac{1}{2}}y-\frac{1}{2}r^{\frac{1}{2}}y'+\frac{3}{2}r^{1/2}y'+r^{3/2}y''$

$-\frac{1}{4}r^{-\frac{1}{2}}y+r^{1/2}y'+r^{3/2}y''+\lambda r^{3/2}y-\mu r^{-1/2}y \quad \Big|\cdot r^{1/2}$

$-\frac{1}{4}y+ry'+r^2y''+\lambda r^2y-\mu y=0$

$r^2y''+ry'+\lambda r^2y-(\mu+\frac{1}{4})y=0$

$\mu+\frac{1}{4}=n(n+1)+\frac{1}{4}=n^2+n+\frac{1}{4}=(n+\frac{1}{2})^2$

т.е. получили ур-ие Бесселя порядка $n+\frac{1}{2}$

$y=J_{n+\frac{1}{2}}\left(\frac{\xi^m_{n+\frac{1}{2}}r}{r_0}\right)$, где $J_{n+\frac{1}{2}}(\xi^m)=0$

$\lambda=\left(\frac{\xi^m_{n+\frac{1}{2}}}{r_0}\right)^2$

$R(r)=r^{-1/2}J_{n+\frac{1}{2}}\left(\frac{\xi^m_{n+\frac{1}{2}}r}{r_0}\right)$

$$u(r,\varphi,\theta,t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=1}^{\infty}J_{n+\frac{1}{2}}\left(\frac{\xi^m_{n+\frac{1}{2}}r}{r_0}\right)r^{-1/2}P_n^k(\cos\theta)\cdot$$

$$\cdot(a_{nk}\cos k\varphi+b_{nk}\sin k\varphi)\cdot C_{mn}\cdot e^{-\lambda_{mn}a^2t}$$

$$u(r,\varphi,\theta,0)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=1}^{\infty}J_{n+\frac{1}{2}}\left(\frac{\xi^m_{n+\frac{1}{2}}r}{r_0}\right)r^{-1/2}P_n^k(\cos\theta)(a_{nk}\cos k\varphi+b_{nk}\sin k\varphi)C_{mn}=$$

$=\frac{3}{2}\sin 2\theta\cos\varphi\cdot J_{5/2}\left(\frac{\xi^1_{5/2}r}{r_0}\right)\cdot r^{-1/2} \qquad (r_0=b)$

$\Rightarrow b=0$, при $k=1,\ n=2,\ m=1\quad C_{mn}\neq0,\ a_{nk}\neq0$

$$P_2^1(\cos\theta) = \frac{3}{2}\sin(2\theta)$$

$$\Rightarrow a_{nk} \cdot C_{mn} = 1$$

Ответ: $$\boxed{u(r, \varphi, \theta, t) = \frac{3}{2}\sin(2\theta)\cos\varphi \cdot J_{5/2}\left(\frac{\mu_2^{(1)} r}{6}\right) r^{-1/2} \cdot e^{-\left(\frac{a\mu_2^{(1)}}{r_0}\right)}}$$

где $\mu_2^{(1)}$ – 1-ый корень $J_{5/2}(x) = 0$.